

APRIL/MAY 2025

23PMA22 — REAL ANALYSIS - II

Time : Three hours

Maximum : 75 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL questions.

1. Show that the set $[0, 1]$ is not countable.
2. Define measurable sets.
3. Define the integral of a non negative function.
4. Define Lebesgue integral.
5. Write the Fourier Series of a function relative to an orthonormal system.
6. State Dirichlet's test.
7. Define Directional derivative.
8. Write the matrix form of the chain rule.
9. Define Saddle Point.
10. When can we say that f is continuously differentiable.



SECTION B — ($5 \times 5 = 25$ marks)

Answer ALL questions.



11. (a) If E_1 and E_2 are measurable, then to show that $E_1 \cup E_2$ is measurable.

Or

- (b) Let $\langle E_i \rangle$ be a sequence of measurable sets. Then $m(UE_i) \leq \sum mE_i$. If the sets E_n are pairwise disjoint, then $m(UE_i) = \sum mE_i$

12. (a) Let f be defined and bounded on a measurable set E with mE is finite. In order that

$$\inf_{f \leq \psi} \int_E \psi(x) dx = \sup_{f \geq \phi} \int_E \phi(x) dx, \text{ for all}$$

simple function ϕ and ψ , it is necessary and sufficient that f be measurable — Prove.

Or

- (b) State and prove Lebesgue Conveyance theorem.

(c) If $f \leq g$ a.e., then prove that $\int_E f \leq \int_E g$.

(d) If A and B are disjoint measurable sets contained in E , then prove that $\int_{A \cup B} f = \int_A f + \int_B f$.

18. State and prove Jordan's theorem.

19. State and prove chain Rule.

20. State and prove the second - derivative test for extrema.



13. (a) Assume $\{\phi_0, \phi_1, \dots\}$ is orthonormal on I . Let $\{C_n\}$ be any sequence of complex numbers such that $\sum |c_k|^2$ converges. Then prove that there is a function (a) f is $L^2(I)$ such that (a) $(f, \phi_k) = c_k$ for each $k \geq 0$, and $\|f\|^2 = \sum_{k=0}^{\infty} |c_k|^2$.

Or

(b) Let f be a real valued and continuous on a compact interval $[a, b]$. Then prove that for every $\epsilon > 0$ there is a polynomial P (which may depend on ϵ) such that $|f(x) - p(x)| < \epsilon$ for every x is $[a, b]$.

14. (a) Let u and v be two real-valued functions defined on a subset s of the complex plane. Assume also that u and v are differentiable at an interior point C of s and that the partial derivatives satisfy the Cauchy-Rieman equations at c , then prove that the function $f = u + iv$ has a derivative at c . Moreover, $f'(c) = D_1 u(c) + i D_1 v(c)$

Or

- (b) Let S be an open connected subset of R^n and let $f: S \rightarrow R^m$ be differentiable at each point of S . If $f'(c) = 0$ for each c in S , then prove that f is constant on S .

15. (a) Let A be an open subset of R^n and assume that $f: A \rightarrow R^n$ has continuous partial derivatives $D_j f_i$ on A . If $F_f(x) \neq 0$ for all x in A , then prove that f is an open mapping.

Or

- (b) Let f be a real-valued function with continuous second-order partial derivatives at a stationary point a in R^2 . Let $A = D_{1,1} f(a)$, $B = D_{1,2} f(a)$, $C = D_{2,2} f(a)$ and let $\Delta = \det \begin{bmatrix} A & B \\ B & C \end{bmatrix} = AC - B^2$. Then prove that (i) If $A > 0$ and $\Delta < 0$, f has a relative minimum at a . (ii) $\Delta > 0$ and $A < 0$, f has a relative maximum at a . (iii) If $\Delta < 0$, f has a saddle point at a .

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. Let E be a given set. Then prove that the following five statements are equivalent.

- E is measurable
- Given $\epsilon > 0$, there is an open set $O \supset E$ with $m^*(O - E) < \epsilon$.
- Given $\epsilon > 0$, there is a closed set $F \subset E$ with $m^*(E - F) < \epsilon$.
- There is a G in $\mathcal{G}_\delta$ with $E \subset G$, $m^*(G - E) = 0$
- There is an F in $\mathcal{F}_\sigma$ with $F \subset E$, $m^*(E - F) = 0$.

17. Let f and g be integrable over E . then prove that

- The function c_f is integrable over E and $\int_E c f = c \int_E f$.
- The function $f + g$ is integrable over E , and $\int_E f + g = \int_E f + \int_E g$.

